COUNTING VS. MEASURING: REFLECTIONS ON NUMBER ROOTS BETWEEN EPISODEMEOLOGY AND NEUROSCIENCE

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Abstract. Pointing on some neuroscience results, concerning the presence in our brain of two distinct prelinguistic systems for processing numbers, we compare different views on the cognitive roots of numbers, in epistemology, psychology and math education. The conclusion is that the two distinct processes of counting and measuring should be better integrated in early math education. To this purpose, an example is reported and commented, of a revealing cognitive behaviour of very young children.

PROLOGUE
A K-degree class episode. The 4 year-old children are engaged in an activity to discover how many days in a week are school-days. Of course, at the beginning their counting competence is very poor. So, how to count the passing days? The idea is to put “something” into a box every day, and then to count these things instead than days. Their first attempt is to put a handful of construction blocks every day, but on Friday they realize that this way is quite inadequate. After a long trouble, they decide to put each day the same “quantity” into the box, namely two construction blocks. Then, on Friday, they turn the box over and count the blocks “two by two”.

During the activity, Camilla says: “Let’s take all the week days and put two of them away”. Notice that a "week train" , like in Figure, with five red wagons (school days) and two yellow wagons (holidays) is standing on the class wall.

INTRODUCTION
Recently, the research domain of neurosciences is expanding at extraordinary speed, and nowadays the need for taking into account its results is widely recognized by educational researchers. For example, in (Tall, 2004), the author identifies these researches, from those concerning innate numerical competencies to those on subjects engaged in complex cognitive tasks, as an emerging strand in cognitive theory. Only two years later, Campbell comes to imagine the birth of a new area of educational research, the educational neuroscience, “that is both informed by the results of cognitive neuroscience, and has access to the methods of cognitive neuroscience” (Campbell, 2006, p. 260).


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Our aim is less ambitious, as we only try to utilize some neuroscience results in order to interpret learners’ cognitive behaviours and to design didactic mediations that really rely on learners’ cognitive resources, and, more generally, to partially re-read or revise classical cognitive models. As a first step in this direction, in a previous paper (Iannece et al., 2006) we have shown how some Houdé’s results based on brain imaging research suggest a critical reflection on some assumptions of the classical cognitive psychology concerning logical thinking. Here, we face one of the more debated question: the birth and the development of the concept of number at school. For the number concept we refer to a structural view of numbers, in the sense of (Sfard, 1991), and to the “number sense” as described in (Arcavi, 1994).

The existence of two “primitive” systems for processing quantities, revealed by neurophysiology studies, suggests to re-think how the number concept develops in children. After a short description of the two systems, we propose a comparison of some of the principal cognitive and epistemological theories about numbers, in order to show how their views correspond to one or another of the two core systems. Then, we argue about the possibility/necessity that the two natural ways of processing numbers, rooted in our brain systems, develop together, thanks to a careful didactic mediation, as two sides of the same coin. To this purpose, in the last section we present a gestalt schema, where the two aspects of numbers and some of their fundamental properties are captured in a unified way; then we illustrate how such a schema can work as a semiotic mediator for a structural grasping of numbers, and, moreover, how the awareness of the brain way of working can be a powerful tool in the hands of teachers to understand and guide their pupils.

THE DIALECTICS BETWEEN THE DISCRETE AND CONTINUOUS ASPECTS OF NUMBERS

The discussion about the cognitive roots of numbers develops along our whole cultural history, involving several different research domains. In the sequel, starting from some neurophysiology results, through a short analysis of some epistemological and psychological theories, we argue that consolidated views about the process of acquisition of the number concept can and should deserve further reflections.

Neurophysiology

Nowadays, it is widely acknowledged that in the human brain, but also in the brain of many superior animals, there are two distinct systems for processing numbers (see, e. g., Feigenson et al., 2004). Within an evolutionary model of the brain (Changeux, 2002), both these core systems are pre-linguistic resources, developed along mankind history, via an “epigenetic” process, as effective tools for interpreting and acting on the external world, in order to guarantee the survival of the human species.

The first system is specialized in recognizing the numerosity of small groups of objects (say, up to four), by the so-called “subitizing”, while the second one provides
“an analogical representation of quantities, in which numbers are represented as distributions of activation on the mental number line” (Dehaene, 2001, pp. 10-11). What is specially interesting is that the second system is activated not only for comparing and manipulating continuous quantities, but also for perceiving and processing discrete quantities in an approximate way.

Indirect evidence of the existence of this sort of “mental line” rests on a big amount of observations (ranging from animals, to babies without linguistic competencies, to adults normally able in symbolic performances, to pathological cases) of two special phenomena: the distance effect and the size effect. In the words of Dehaene, “the distance effect is a systematic, monotonous decrease in numerosity discrimination performance as the numerical distance between the numbers decreases. The size effect indicates that for equal numerical distance, performance also decreases with increasing number size. Both effects indicate that the discrimination of numerosity, like that of many other physical parameters, obeys Fechner’s Law.” (ibid., pp. 6-7). According to a Dehaene’s metaphor, our “mental number line” works like an accumulator: that is, a schema that both allows a static comparison of quantities and identifies numbers as dynamic results either of storing separate arbitrary units or of adding/subtracting two approximate quantities. A picture like the following

_well captures, in our opinion, the fundamental properties of the “accumulator”. In fact, it shows that:

  a) the additive structure is embodied in number sense;
  b) counting and measuring are two strongly intertwined processes;
  c) the cardinality of any state and the effects on it of additive transformations are simultaneously recognized;
  d) the mental number line is nonlinearly compressive, with pairs of numbers lying closer together as their magnitude increases.

To sum up, we are provided with two different sources of number sense, two irreducible perceptive “moves”, that can be contrasted with two complementary aspects of the reality, the discreteness of objects and the extension of magnitudes. To

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2 Recently, a suggestive hypothesis has been formulated (Walsh, 2003), namely that the second brain system is based on the same mental circuitry that has been elaborated by the human species to perceive and conceptualize time and space (the “locus of a common magnitude system”).

3 Also known as Weber-Fechner’s Law, states that the magnitude of a sensation is proportional to the logarithm of the intensity of the stimulus causing it. It has been recognized also in the discrimination of numerosity since many years (see, e. g. (Changeux & Dehaene, 1993)).

4 According to an evolutionary model of the brain, that is our philosophical option, it is almost obvious that our mental structures appear adequate, not to say “isomorphic”, to reality structures.
trust these results means to recognize that at the origin of the number concept there are two distinct but correlated counting processes: counting discrete objects as a linguistic evolution of subitizing and counting by measuring. Moreover, the latter would be predominantly activated, because it is language-independent and is spontaneously used in everyday experience to evaluate distance, time and so on.

Psychology and Math Education

As said before, the cognitive roots of numbers have been investigated from several points of view. Psychological studies, form Piaget onward, as well as math education studies, prevalently assign, with minor differences, a primitive and priority role to natural numbers, due to the original action of counting. But it is also possible to recognize a different approach, in such a way that we could speak of a duality which resembles that observed at a neuronal level.

Here we refer, for the first approach, among others, to (Sfard, 1991) and (Lakoff & Núñez, 2000). In her cognitive reconstruction of the number concept within a dialectic process/object, Sfard proposes a schema where the counting process constitutes the starting point, whereas the measuring process appears only in a later step, when rational numbers are generated. On their part, also Lakoff & Núñez focus on natural numbers, since they assume the subitizing as the root for number concept. Then they utilize four grounding metaphors to build the whole arithmetic. One of these metaphors (the measuring stick metaphor) relies on spontaneous activities of measuring that could allow to introduce a wider range of numbers.

A different approach is followed by V. V. Davydov. In (Davydov, 1982) the genesis of the number concept is rooted in the experience of measuring continuous quantities. The notion of quantity comes from comparison of elements of a given class (e. g., lengths of segments, amounts of water, weights, etc.), while measuring means to relate a given quantity with a part of it, assumed as a unit. Counting itself may be conceived as the particular measuring process of discrete objects, whence the sequence of natural numbers appears as just an example of quantity. But a deeper acquaintance with quantities allows children to enlarge their knowledge of numbers to include integers, rationals and reals.

Therefore, Davydov suggests that the practice with the properties of the quantities should precede in the early education the practice with natural numbers. He proposes activities where, once recognized and expressed an order relationship between two
quantities, the attention is focused on the quantity that has to be added to the smaller to obtain the larger one. He suggests also to transform a generic quantity into a segment, as an effective “intermediate strategy of graphic representation” (a semiotic mediator, in Vygotskijan words), something like the picture aside, where the crucial point is to recognize that \( X = B - A \). The space metaphor allows to visualize the relationship between the two quantities A and B and to interpret subtraction as the formal description of the process of comparing A and B, rather than as a decrease: in a sense, thanks to the representation adopted, the object itself and its symbolic expression “relate directly to the properties of the object. In a school subject, intermediate means of description have crucial significance because they mediate between a property of an object and a concept.” (Davydov, 1982, p. 237).

A similar view is proposed by Gelman & Gallistel, who point on neuroscience results. The authors identify the real numbers as closest to the originary perception of space: “The evidence from experiments that probe the properties of numerical representations in non-verbal animals and humans suggest that there exists a common system for representing both countable and uncountable quantity by means of mental magnitudes formally equivalent to real numbers. These mental magnitudes are arithmetically processed without regard to whether they represent countable or uncountable quantity. […] Then the real numbers are the psychologically primitive system, not the natural numbers. The special role of the natural numbers in the cultural history of arithmetic is a consequence of the discrete character of human language, which picks out of the system of real numbers in the brain the discretely ordered subset generated by the nonverbal counting process, and makes these the foundation of the linguistically mediated conception of number” (Gallistel et al., 2006, p. 270).

**History and Epistemology**

Depicting an outline of the history of numbers, Sfard says “much time elapsed before mathematicians were able to separate numbers from measuring process and to acknowledge that the length of any segment represent a number even if it cannot be found in the ‘usual’ way.” (Sfard, 1991, p. 12). In the sequel, we will try to read in a slightly different way the history of numbers, recognizing in it the effort, rather than to separate numbers from measures, to integrate two complementary aspects of numbers. In our opinion, this is a warning that a similar effort has to be made in the learning process in order again, not to separate, but to knowingly integrate them.

The dichotomy discrete/continuous has been pervasive in the whole western cultural history. Here we claim that in the history of mathematics numbers are hardly separable from geometry foundations. We can recognize at the origin of civilization two independent mathematical needs: measuring space and counting discrete objects. The Pythagorean school thought that natural numbers (and their ratios) could satisfy not only the latter but also the former need, until the discovery of incommensurable segments destroyed this claim. As a consequence, numbers and geometry divorced,
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and Euclid could build his monumental “geometry without numbers”. For more than two thousand years arithmetic and geometry were treated as separate domains; but the need of conceiving numbers as measuring, not only as counting tools, slowly re-emerged. It is interesting to report a passage from Newton’s *Arithmetica Universalis*: “Per numerum non tam multitudinem unitatum, quam abstractam quantitatis cuiusvis ad aliam eiusdem generis quantitatem, quae pro unitate habetur, rationem intelligimus.⁵

(Newton, 1761, p. 2). The XIX century ultimate stage of this process, the so called Arithmetization of Calculus, culminated with the construction of real (and complex) numbers⁶ in terms of natural numbers, by means of successive steps: the hierarchy today predominant in the formal presentation of the various kinds of numbers; also referred to by Sfard in her schema. But in our opinion, this formal reconstruction, if succeeds in reducing several notions to one, doesn’t exactly correspond to the cognitive roots of the various kinds of numbers. The priority given to natural numbers, either conceived as members of a special sequence (Peano) or as cardinals of finite sets (Cantor), emphasizes the discrete side of numbers, putting in shadow the continuous one. In other words, the reconciliation between arithmetic and geometry is successfully accomplished from a formal point of view, but is incomplete from a cognitive one.

It is perhaps noteworthy to notice that there is also a different formal approach to numbers. In fact, A. Kolmogorov, and other mathematicians, among which H. Lebesgue, following Newton’s precursory idea⁷, reversed the traditional path, giving priority to the domain of real numbers, wherein the rationals, the integers and the natural numbers are viewed as particular subsets.

So, also at an epistemological level, one could consider the two above approaches as the formal counterparts of the two primitive cognitive attitudes of our brain.

**DISCUSSION AND CONCLUSIONS**

We agree with Sfard in recognizing a repeated dialectics between processes and objects in the development of numbers. But we think that the relationships between the various steps are less linear than in her schema: namely, we recognize at the very beginning not only the process of counting, but also that of measuring: so to say, two legs for number sense-making.

The short analysis presented in the previous Section, about psychological, educational and epistemological views, shows that, in all cases, there is a choice of one of the two horns, discrete and continuous, with perhaps a predominance of the former, as primitive root of the number concept. In the wider frame provided by

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⁵ “By a *number*, rather than a multitude of units, we intend the abstract ratio of any quantity to another one of the same kind, assumed as unit”. (Our translation)

⁶ It is not a case that the crucial idea in this process is Dedekind’s translation of a geometrical property, the continuity of the line, into a numerical one, for rational cuts.

⁷ The idea of real number as a ratio of two magnitudes has been also contended by G. Frege in his *Die Grundlagen der Arithmetik*, as a criticism to Dedekind and Cantor’s theories.
neurophysiology studies, the two alternatives, corresponding to the two basic neural systems for perceiving and elaborating quantities, appear as two sides of the same coin. Just as the two brain systems cooperate to support our understanding and our action in the world, we believe that the two corresponding aspects of numbers should be developed together. According to Davydov’s perspective, “the ultimate aim of instruction in mathematics should be clear from the very beginning” (Davydov, 1982, p. 230), the integration should be pursued in math education when firstly dealing with natural numbers. In fact, the counting process, starting from subitizing, rapidly develops thanks to the discrete feature of the language, allowing, after successive steps, the exact manipulation of rational and real numbers. At the same time, the measuring process helps to immediately perceive natural numbers as particular members of a wider numerical domain, favouring a structural view of numbers and, moreover, induces an early development of abilities recognized in (Arcavi, 1994) as markers of “number sense”, like the ability to approximately estimate a given quantity, etc..

For these reasons, a crucial educational problem is how to design and realize an early and effective strategy of cultural mediation, able to create solid links between the discrete and continuous aspects of numbers. The schema reported in the figure aside, that generalizes and synthesizes all the previous schemata, seems to be a useful tool in this direction. Here, the discrete structure (the bullets in figure) induces the choice of a unit of measure, that uniformly subdivides the line, rectifying the nonlinear compression of the accumulator. On the other hand, the correspondence between the discrete structure and the discretized one projects onto the former the structural properties, pointed out by Davydov for continuous quantities. Therefore, the schema represents not only a semiotic mediator of the cognitive interference of the discrete and continuous aspects of numbers, but also a resonance mediator, in the sense of (Guidoni et al., 2005) between natural cognitive resources and disciplinary contents, in particular the additive structure. It is a useful tool in the hand of teachers both to interpret pupils’ behaviours and to plan class activities.

Let us now go back to our Prologue. We focus our attention on two points:

a) the children spontaneously choose two blocks as a unit of measure;

b) Camilla’s reasoning follows the path: “consider all, and take two away”.

Well, in both behaviours we can see that what is working is the “mental number line” system, since a global perception of quantity seems to prevail over the use of one-one correspondences, typical of the counting process. In particular, at the basis of Camilla’s reasoning, it is recognizable the above schema, expressed in her experience by the week train. The fact that she autonomously utilizes the schema while engaged in a problem solving situation, seems to us a strong marker of its validity.
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A final remark concerns the teacher’s reaction. In her previous similar experiences she had forced children toward the correspondence one day-one block. This time, she was firstly astonished, but then, adopting the point of view suggested by neurophysiology studies, thanks to a training path within a research team, she realized that her pupils were anyhow building their number sense, even if not in the “usual” way.

REFERENCES


